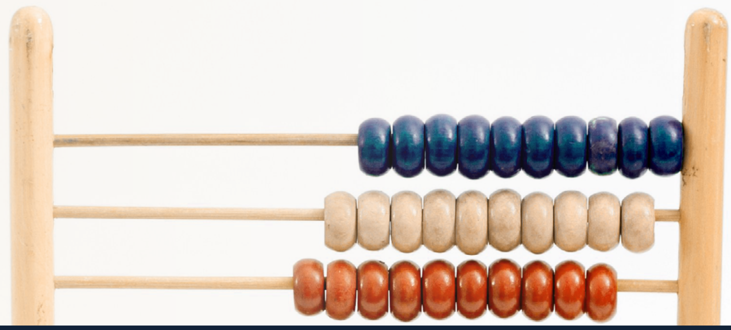
New York State
EDUCATION DEPARTMENT
Knowledge › Skill › Opportunity



Brief 6:

The Role and Challenges of Using Representations

Supporting Evidence-Based P–12 Mathematics Teaching Practice

Produced for the New York State Education Department
by Deborah Loewenberg Ball and TeachingWorks at the University of Michigan

> Instructional programs from prekindergarten through grade 12 should enable all students to: (a) create and use representations to organize, record, and communicate mathematical ideas; (b) select, apply, and translate among mathematical representations to solve problems; (c) use representations to model and interpret physical, social, and mathematical phenomena.
>
> (National Council of Teachers of Mathematics, 2000, p. 67).

Representations: Central Resources and Practices in Mathematics

In mathematics teaching and learning, representations are central resources in mathematical reasoning, problem-solving, and communicating for both students and teachers. "Representations" refers to several different kinds of models and materials important in mathematics itself and in support of mathematics learning (Lesh et al., 1987):

- **Physical:** e.g., manipulative materials such as Unifix cubes, base ten blocks, algebra tiles
- **Visual:** e.g., diagrams, drawings, graphs
- **Symbolic:** e.g., numbers, expressions, equations
- **Verbal:** e.g., explanations and discussions
- **Contextual:** e.g., real-world situations and problem-based contexts

Representations allow teachers and students to work with abstract mathematical ideas in a range of ways. In teaching, each of these types of representations can highlight different aspects of the mathematical ideas, helping to support student learning. Physical representations, often referred to as concrete representations or manipulatives, allow students to physically move objects, make trades for equivalent objects, and rearrange objects to help them to solve problems. Some physical objects maintain key mathematical features they are trying to represent. For example, base ten blocks and bean sticks show ten as one object made of ten individual objects connected together. Other physical objects are more abstract. Place value disks are simply colored chips representing each of the place values but do not make visible the differences in quantity or the concept of grouping fundamental to place value.

Visual representations more permanently represent the mathematical ideas, but like physical representations can be more or less abstract. For example, multiplication arrays closely represent the individual objects being multiplied as well as the product, but area models make clear that the answer is an area that is a product of lengths. Although there exists a debate around the order in which these representations should be used in learning, there is agreement that engagement with them is necessary for conceptual understanding and meaningful use of procedural skill.

Choosing and building representations are also core to doing mathematics. As explained by the National Council of Teachers of Mathematics (2014), "When students learn to represent, discuss, and make connections among mathematical ideas in multiple forms, they demonstrate deeper mathematical understanding and enhanced problem-solving skills" (p. 24). Representation is not only an external process involving, for example, diagramming, drawing, and explaining. It is also an internal process, as students develop rich mental models of mathematical concepts over time through making sense of and using multiple representations. Choosing and using appropriate representations is a key part of mathematical fluency and problem solving.

The Work of Using Representations in Teaching Mathematics

Using representations with care requires a repertoire appropriate for particular mathematical content and specialized mathematical understanding of how specific representations can highlight crucial aspects of the math. Equitable and effective mathematics teaching engages students in creating, interpreting, using, and making connections across representations. Students are given opportunities both to exercise agency in using representations that make sense to them and to learn to use new representational forms to deepen their understanding (Taylor & Dyer, 2014). Representations are central in providing access to mathematics for students with diverse background knowledge and approaches. In particular, research has demonstrated that multilingual learners, students with special needs, and those struggling with a concept are better able to engage in mathematical discourse when they are able to gesture and show their thinking with the supports of manipulatives and visual representations (Fuson & Murata, 2007).

Additionally, the classroom environment must be welcoming and supportive of the broad use of representations which can support engagement with and understanding of rigorous mathematics. A welcoming and affirming mathematics classroom creates an environment where students can comfortably use the mathematical tools at their disposal to make sense of mathematical ideas, share their thinking, and justify their reasoning. Without this welcoming environment paired with an emphasis on symbolic reasoning, students are likely to see manipulatives and representations as "less than" symbolic reasoning rather than a powerful tool for sense-making. For more information on creating a welcoming culture and fostering high expectations, see the [Four Principles of Culturally Responsive-Sustaining Education.]

Spotlight on Research

As students interact with representations in conversation with one another, they authentically and organically create "bridging language"—transitional language that bridges the language of specific problem contexts with official mathematical language. For example, in examining and discussing tables and graphical representations of slope, students generated language like "it increased by X each time." These various forms of language mutually reinforce students' understanding of the underlying mathematical concepts while also developing official mathematical vocabulary (Herbal-Eisenmann, 2002).

Using Representations: What Teachers Need to Know and Be Able to Do

The National Council of Teachers of Mathematics (2014) guidance identifies using and connecting mathematical representations as one of eight core mathematics teaching practices. In order to use representations effectively in instruction, teachers need to understand, among other things, the instructional and mathematical affordances and constraints of different representations in the service of learning goals. Opportunities to explore and compare different representations and materials is a useful focus for professional learning. Using them well also depends on deliberate mathematical consideration of specific models, contexts, materials, or other forms. Selecting and adapting tasks that prompt students to interact with representations and to create their own representations to solve problems matters. All of this requires teachers to build a flexible repertoire of practices for supporting students to work with and make sense of representations.

Preparing for instruction using representations

In using curriculum materials to prepare for instruction, teachers attend not only to the mathematical goals of a unit or lesson but also to the set of representational resources provided to support those goals. Teachers must weigh the strengths and limitations of different representations based on the mathematics they make available. Teachers' decisions are also informed by what they know about their students' previous experiences and the challenges students are likely to encounter. For example, which models will feel familiar to students and which ones may need to be more formally introduced? How will students likely interact with a set of manipulatives for the first time? Teachers must translate this understanding of their students and the relative affordances of different representations into their planning.

Mathematical tasks vary in the representational demands they place on students. Some tasks are designed for students to use particular manipulatives or models to develop conceptual understanding. Some tasks invite students to be critical consumers of data representations. Other tasks allow students to invent or choose a representation that supports their problem-solving. Teachers consider these task demands as they make decisions about when to adapt them and how to implement them with their students and mathematical purposes in mind.

Using representations during instruction

As students engage in mathematical tasks, teachers encourage sense-making by providing time for students to explore and experiment with representations as they solve problems, offering support when needed. This support may sometimes include teaching students how to use a particular manipulative (e.g., algebra tiles) and setting up the language and routines for their productive use. Making connections across representations is crucial, and although teachers may explicitly show how one representation maps onto another, it is also important to support students in developing both the disposition and skill to do so themselves. Teachers may model this skill by asking questions to elicit student thinking while deliberately recording those ideas on the board using a combination of symbolic representations, models, and annotations (Garcia et al., 2021). This representational work supports students in making sense of one another's ideas and models for students how to use representations to communicate about mathematics.

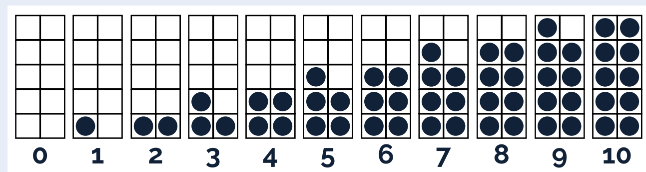
To help students make connections across representations, teachers decide how to select and sequence student work to present a range of representations and strategies (Smith & Stein, 2011). They take care to use board space strategically in the service of collective sense-making (Leatham et al., 2023) and they ask questions that press students to look across representations to make comparisons and notice connections.

Learn More about Student Representations

For examples of the kinds of questions teachers can ask along with a vignette of a classroom discussion, check out [Mathematical Representations: A Window into Student Thinking,] a blog post on the website of the Institute for Learning at the University of Pittsburgh by Amaru Pareja, a sixth-grade teacher in Syracuse.

Spotlight on Research

As Christine Losq (2005) clearly demonstrates in her article "Number Concepts and Special Needs Students: The Power of Ten-frame Tiles," not all representations are created equal. There are some crucial affordances of ten-frame tiles that make them a good representational choice for teachers seeking to foster students' counting and computation skills. Ten-frame tiles show a unique image for each number from 0 to 10.



The 5 x 2 array structure of the ten-frame allows each number from 0 to 10 to have a unique picture. This structure supports children to subitize quantities as they come to "internalize the 'shape' of a quantity in instantly recognizable pictures." Unlike cube trains or base ten blocks, the dots on ten-frame tiles are "clearly separate and countable." Additionally, children can see each represented quantity in a range of ways, supporting flexibility and number sense.

Spotlight on Research

Number lines are a powerful representation that support both number sense and operations. In their article "Building a Strong Conception of the Number Line," John Lannin, Delinda van Garderen, and Jessica Kamuru (2020) explain how carefully designed task selection can help students to build and refine their own conceptions of the number line, including the spatial relationship between numbers. They share case studies of children's number line understanding, examining how their understanding changed as they engaged with a series of tasks. They argue that questions such as "what friendly numbers are close to [a particular number]," "what friendly numbers are far away from [a particular number]," and "how are these numbers on the number line related" can keep the focus on big mathematical ideas while refining number line conceptions. When working with number lines, teachers should keep in mind that the number line is not only a representation that can support learning, but is also itself an important mathematical object in later mathematics.

Spotlight on Research

The Class Party Problem

Ms. Simpson formed a committee of students to investigate sites for a class party. The committee does not know how many of the students will attend and has not yet decided whether each class member can invite a guest. However, they have brought back the following details on parties:

- Water World • Swimming, hot dog, chips, and drink. Cost: $100 to reserve the pool and $5 per person.
- Pizza Pi and MoviePlex • Pizza, drink, movie. Cost: $10 per person.
- Skate Til Late • Skating and ice cream (eat before you come). Cost: $200 to rent the skating rink and $2 per person for skate rental.

price is the primary focus of your decision, which party otion is the best?

ow would you convince other class members of this?

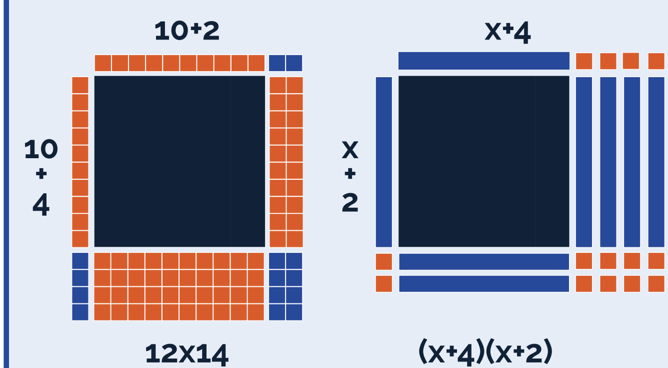
ow would you advise other groups planning a party, so iat they could make a decision, no matter how many uests they had?

g. 1 The Class Party problem

When students are supported to make decisions about the usefulness of various representations, they not only can deepen their mathematical understanding but also can develop a sense of agency. In "Representation as a Vehicle for Solving and Communication," Ron Preston and Amanda Garner (2003) illustrate the power of designing and using tasks like the "Class Party Problem" to engage students in selecting, using, communicating about, and evaluating a range of representations to solve a real-world problem. The task prompted 7th grade pre-algebra students working in small groups to use all of the following representations: word rules, informal and formal language, tables, graphs, equations, patterns, and hand gestures. Through discussion of the various groups' solution strategies, students made connections across representational forms and evaluated the advantages and disadvantages in relation to the problem context. Check out their classroom vignette in the article for more on how students engaged with the task and one another's solutions. They also provide a useful table summarizing some of the typical uses, advantages, and disadvantages of different representations, and share insights on teaching dilemmas that arise when inviting students to choose their own representations.

Spotlight on Research

Representations can support students to experience mathematics as a coherent interconnected system of concepts. In their article "The Area Model: Building Mathematical Connections," Alyson Lischka and Christopher Stephens (2020) illustrate the power of manipulatives and rectangular area models to do just that. Using examples from across grade levels, they demonstrate how manipulatives and area models can support the teaching and learning of a range of concepts involved in multiplicative reasoning with whole numbers, from fractions and binomial expressions in algebra to derivatives in calculus. "Students who learn to visualize the numerical application of the distributive property in this way [using area models]," they report, "build a foundation for considering multidigit, fraction, and binomial multiplication with an area model" (p. 189).



Multiplication using base 10 blocks

Multiplication using algebra tiles

Reflect & Analyze:

1. How do you currently support students to invent, use, compare, and connect representations? Were there aspects of using representations in the mathematics classroom that you hadn't thought about before?

2. Which representational forms would you say your students are most comfortable with? Which are you most comfortable with? Why? How does that impact instruction in your classroom?

3. What would you like to learn more about in order to feel more confident using different representations in your classroom? Who in your context could help provide resources for you with this? How might you work with grade-level or building colleagues to learn more about representations?

Key References

Fuson, K. C. & Murata, A. (2007). Integrating NRC principles and the NCTM Process Standards to form a class learning path model that individualizes within whole-class activities. *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership, 10*(1), 72–91.

Garcia, N., Shaughnessy, M., & Pynes, D. (2021). Recording student thinking in a mathematics discussion. *Mathematics Teacher: Learning and Teaching PK–12, 114*(12), 926–932.

Herbel-Eisenmann, B. (2002). Using student contributions and multiple representations to develop mathematical language. *Mathematics Teaching in the Middle School, 8*(2), 100–105.

Lannin, J. K., van Garderen, D., & Kamuru, J. (2020). Building a strong conception of the number line. *Mathematics Teacher: Learning and Teaching PK-12 MTLT, 113*(1), 18–24.

Leatham, K., Peterson, B. E., Freeburn, B., Graff, S. W., Van Zoest, L. R., Stockero, S. L., & Kamlue, N. (2023). Using public records to scaffold joint sense making. *Mathematics Teacher: Learning and Teaching PK–12, 116*(11), 856-864.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 21, 33–40). Lawrence Erlbaum.

Lischka, A. E., & Stephens, D. C. (2020). The area model: Building mathematical connections. *Mathematics Teacher: Learning and Teaching PK–12, 113*(3). 186–195.

Losq, C. S. (2005). Number concepts and special needs students: The power of ten-frame tiles. *Teaching Children Mathematics, 11*(6), 310–315.

Midgett, C. W., & Eddins, S. K. (2001). NCTM's Principles and Standards for School Mathematics: Implications for administrators. *NASSP Bulletin, 85*(623), 35–42.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. National Council of Teachers of Mathematics.

Preston, R. V. & Garner, A. S. (2003). Representation as a vehicle for solving and communication. *Mathematics Teaching in the Middle School, 9*(1), 38–43.

Smith, M., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematics discussions*. National Council of Teachers of Mathematics.

Taylor, E. V., & Dyer, E. (2014). Teacher goals and dilemmas in the use of mathematical representations. *Mathematics Teacher Educator, 2*(2), 171–184.