New York State EDUCATION DEPARTMENT
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Brief 3:

High-Leverage Mathematical Content

Supporting Evidence-Based P–12 Mathematics Teaching Practice

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> All young [people] must learn to think mathematically, and they must think mathematically to learn.
>
> (National Research Council, 2001, p. 1)

What is meant by "high-leverage mathematical content"?

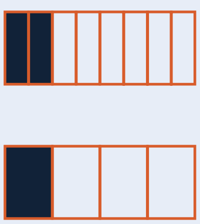
High-leverage mathematical content comprises mathematical concepts, skills, and practices essential for mathematical competence. It focuses on what it takes to be able to use mathematics to solve problems and to reason about questions and situations across a range of settings. Mathematics includes arithmetic, algebraic computation, and estimation, but it also encompasses a vast domain of human thinking and ideas, developed over centuries in diverse cultures and contexts and for a wide range of uses. Historically, the U.S. school curriculum began by focusing on the measurement and quantitative skills needed for commerce. It has expanded over decades to include many more domains, such as probability (the math involved in figuring out the likelihood of a particular outcome) and discrete mathematics (for example, the math involved in finding the number of ways that a set of objects can be rearranged), and more areas of application (for example, engineering).

The curriculum in this country has been described as a "mile wide and an inch deep," a critique of the lack of depth and focus needed to develop in young people the capacity for mathematical problem solving and thinking. It is also fragmented, with bits of a particular mathematical strand showing up each year and different areas separated in the chapters of instructional materials. Identifying key ideas and practices as high-leverage mathematical content and building connections among them is essential to supporting the development of mathematical competence. The New York State Next Generation Mathematics Learning Standards contribute to this goal by providing coherence links within the standards document to help teachers build connections across grade levels.

Spotlight on Practice

Curriculum materials often do not make connections across grades and mathematical concepts. Teachers can help to build these connections through their instructional choices.

Ms. Natasha teaches fourth grade and is helping her students understand equivalent fractions. The guidance given by her instructional materials focuses on shading rectangles—for example, shading 2/8 of a rectangle to show that it covers the same area as 1/4.



Because she wants to support her students to make connections, Ms. Natasha asks her students for other examples of things they have learned where it was useful to represent something in equivalent form. Several students recall that in carrying out multi-digit subtraction, they learned to write equivalent forms of a number to make it possible to subtract, as in this example, where 81 is written as 7 tens and 11 ones, which is the same value as 8 tens and 1 one:

$$\begin{array}{r} 7\ 11 \\ \cancel{81} \\ -\,38 \\ \hline \end{array}$$

What makes mathematical content “high-leverage?”

High-leverage mathematical content centers on the ideas and concepts that are foundational across grades, in the field of mathematics, and for deep capacity to think and solve problems. It is a combination of mathematical knowledge and mathematical practices, focused on core ideas and ways of thinking, understood in ways that make it possible to use mathematics flexibly to reason, think critically, and solve problems. High-leverage mathematical content is:

- **Inclusive of concepts, procedures, skills, language, and practices.**
- **Fundamental across the early grades and through high school.**
- **Foundational to mathematical reasoning and thinking.**
- **Crucial for accessing and mastering more advanced knowledge and skills in mathematics.**

The idea of high-leverage mathematical content is centered on students, and what they need to learn so that they can use the tools and practices of mathematics and see themselves as capable to do so.

Spotlight on Practice

Mathematics problems that provide opportunities to develop concepts, fluency, skill, and reasoning can support students to practice using mathematics, and to learn crucial practices.

The 8’s Problem is an example of such a problem:

> Write a string of 8’s and insert + signs between them to equal 1000. You can use as many 8’s as you want, and you can also put them together like this: 88 (but not like this: 8/8). How can this be done? How many different ways are there to do this?

One solution is to add 125 8’s because 125 x 8 = 1000. But this is just one of the solutions.

Working on this problem involves computation practice; use of concepts of addition, subtraction, multiplication, and division; seeing and using mathematical structure; methods of recording; skills for persevering; and practices of explaining and reasoning. Comparing the mathematical opportunities of this problem with a common computational practice worksheet highlights the way that high-leverage mathematical content can be developed.

(Adapted from Gelfand & Shen, 1993).

One example of high-leverage content across the grades

Elementary

In elementary grades, students learn the foundational ideas underlying fractions concepts. They learn to think flexibly about fractions as numbers (e.g., 1/2 is one way to represent the number 0.5 and is at the same point on the number line). They also can use fractions to name part-whole relationships (1/2 can be represented by a whole divided into two equal pieces with one shaded) and as relationships (e.g., 1/2 is equivalent to 2/4 because the relationship between the numerator and denominator is the same in both fractions). These ideas are fundamental to developing ratio and proportional reasoning in the middle grades, often serving as a “gatekeeper” to middle school mathematics.

Middle School

In the middle grades, students extend the foundational concepts of fractions to build their understanding of ratio and proportion. They learn to think flexibly about a ratio representing the relationship between two quantities, understanding that these are often not part-whole relationships and are often different units of measurement altogether (e.g., people and work-hours). They understand proportions as equivalent ratios, utilizing their strategies for finding equivalent fractions to solve proportion problems and learning new strategies for situations where these don’t apply. Understanding of ratio and proportion is fundamental to the concept of rates of change in the secondary grades and is a fundamental part of everyday mathematical sensemaking about information.

High School

In late middle school and throughout high school, students extend the foundational concepts of ratio to build understanding of rates of change and their impact on functions. They examine how the relationship between two variables is represented in graphs, tables, and equations. They learn to identify when rates of change are constant or varied and how to identify a best-fit-function that describes the relationship between the variables. These fundamental understandings lay the foundation for both practical applications of mathematics and calculus, which focuses on how functions change.

Why is it important to identify “high-leverage mathematical content?”

The school curriculum has been expanding over decades, sometimes adding more content without subtracting anything. Everything seems important and pacing guides can often undermine the need for deep understanding and capacity with core mathematical ideas. Instructional materials have become longer and more complex over time, with many learning goals and a plethora of additional resources for the teacher and for students. This diffuseness is not easy to navigate. Topics that are related are treated as different. Take, for example, the use of area models with multiplication in fourth grade and geometric representations of binomial multiplication in high school algebra. In the fourth grade, the product of two numbers is represented as the area of a rectangle with those side lengths. Those side lengths are broken down into sums to make the multiplication manageable and the area is calculated for each of the component parts. The same is true for the representation of binomial multiplication. The product of the binomials is the area of a rectangle with those binomial side lengths. While the processes and reasoning are the same for these two ideas, the teaching of binomial multiplication is often treated as new for students and unconnected to prior learning.

$35*24=840$

	30	5
20	$30*20=600$	$5*20=100$
4	$30*4=120$	$5*4=20$

$(3x+5)(2x+4)=6x^2+22x+20$

	3x	5
2x	$3x*2x=6x^2$	$5*2x=10x$
4	$3x*4=12x$	$5*4=20$

Rapid changes in technology and artificial intelligence mean that the adults of this upcoming generation will need skills of sensemaking, critical interpretation, and mathematical reasoning and judgment, and that it will be far less common to perform precise calculations or conversions. For example, adults will not often need to carry out long division with large numbers, but they will need to know whether a result is reasonable. Work on conventional algorithms such as long division could center the high-leverage content of place value, yet too often it is focused on the answer more than on understanding the incredible mathematical structure that underlies the procedure. Naming content that is high-leverage can support students and teachers to focus on developing crucial mathematical competence. Understanding what content is high-leverage and therefore of highest priority can help teachers manage the pressure of content “coverage” and pacing guides, ensuring that crucial content is not missed or too lightly or quickly treated. Supporting teachers to be able to make these determinations can be developed through professional learning.

What are some examples of “high-leverage mathematical content?”

Base-ten system of place value and numeration

One clear example of high-leverage mathematical content is the base-ten system of place value and numeration. How does it work? How is it possible to use only 10 digits and yet be able to write any number? What does this compressed system enable in calculating, estimating, and using number sense? Recognizing the power of this system would mean that more attention would be paid to students’ understanding of the structure and less to identifying what place a particular digit is in. For example, why do we say we “add a zero” when we multiply by ten but we say we “move the decimal point over” when we divide by ten? This example highlights the common fragmentation of connected ideas as well as the lack of focus on what makes place value high leverage.

Spotlight on Practice

What does it mean to “add a zero” to multiply by ten? Take the number 54, for example. 54 x 10 = 540. We are not “adding” anything, much less a zero. The real explanation is that 54 is composed of 5 tens and 4 ones. When we multiply by 10, we now have ten sets of 5 tens and 4 ones. That means we now have 5 groups of 10 tens (or hundreds) and 4 groups of 10 ones (or tens), so we move the 5 from the tens place to the hundreds place and the 4 from the ones place to the tens place. When we do that, the value of the 5 is now 5 hundreds and the value of the 4 is now 4 tens. The structure of the place value system makes it easy to transform the value of a number by where it is located—its place. As we move to the left, we are multiplying by a factor of ten each time. The zero in 540 is because 54 is 5 tens, 4 ones, and 0 tenths. The zero is not holding a place. Like the 5 and the 4, it has simply been moved one place to the left. This is at the heart of a place value system.

Based on this explanation, how would we explain dividing by 10? It is structurally the same. We are not moving the decimal point; instead, we are again moving the numbers, but in this case to the right.

Number line

Another example of high-leverage mathematical content is the number line and its properties as a fundamental mathematical object. Students see number lines from the earliest grades, but they only encounter whole numbers—rarely negative numbers or fractions—until much later. Moreover, the infinities of the line—in each direction and between any two points—is left unexposed. The number line holds all the rational numbers and is crucial for many ideas and kinds of reasoning. This also highlights that fractions when first introduced do not seem to be numbers at all, but merely parts of wholes. Once again, the lack of attention to these ideas as high-leverage leaves them disconnected and less powerful.

Fractions

Fractions are another key example of high-leverage content. Understanding fractions as rational numbers, connected to decimals which are place-value based representations of rational numbers, is foundational to being able to operate and reason with numbers. Fractions as division are the building blocks of rational functions, which are crucial for competence with algebraic skill and reasoning. Being able to represent quantities in equivalent forms for particular purposes is an example of the leverage that fractions afford.

Solving equations

The basic principles of solving equations are also high-leverage content. Understanding solving an equation as isolating the variable in order to determine the value and understanding that isolating the variable involves performing operations to the two equivalent expressions that maintain equivalence are fundamental to advanced work with equations and functions. Students are exposed to equivalence in very early grades as they learn different ways to represent numbers, but the meaning of equivalence (and the equal sign) are often left implicit, resulting in many students viewing the equal sign as an operator that tells you to find the answer rather than a symbol noting equivalence between two mathematical objects.

Likewise, work to solve equations by isolating the variable begins in elementary school. Students solve “missing value” problems where they use reasoning and their understanding of inverse operations to determine the missing value. As students progress, the solving of equations often gets reduced to a disconnected set of cases where students follow a prescribed series of steps without meaning. Teachers can create opportunities to support students in recognizing that there are many possible ways of solving the equation as long as the operation you perform on the expressions maintains the equivalence. Raising this idea up would mean moving away from phrases like “move the number to the other side,” which hides the reasoning behind why this move is appropriate and helps you to isolate the variable.

Core practices from the New York State Next Generation Mathematics Learning Standards

High-leverage mathematical content is more than concepts and procedures. The New York State Next Generation Mathematics Learning Standards specify a set of eight core practices that are fundamental to mathematical competence. Making sense of problems and persevering in solving them is broadly important and crucial, for example, for solving word problems. Being able to explain and justify rules and procedures is highlighted for the power gained from understanding why something works. Reasoning, explaining, and proving are high-leverage because they enable learners critical agency in making sense and solving problems. All eight of the Standards for Mathematical Practice (New York State Department of Education, 2019) are high-leverage because they describe both what students should be doing as they learn and what capacities are essential to being a person who can use mathematics and who identifies as a capable doer of mathematics.

Standards for Mathematical Practice

- Make sense of problems and persevere in solving them
- Reason abstractly and quantitatively
- Construct viable arguments and critique the reasoning of others
- Model with mathematics
- Use appropriate tools strategically
- Attend to precision
- Look for and make use of structure
- Look for and express regularity in repeated reasoning

What Do Teachers Need to Support High-Leverage Mathematical Content?

Teachers need opportunities to develop deep mathematical knowledge because teaching requires specialized mathematical understanding that is fundamental to the work of helping others learn math. Teachers with higher levels of this kind of mathematical knowledge for teaching are more effective in helping students grow mathematically.

Spotlight on Research

Teaching requires understanding the math that students are to learn. But teaching also requires different understanding, different from just being able to do the math. Teaching involves being able to explain why a procedure works and to define mathematical terms in ways that are correct but also accessible.

"The mathematical demands of teaching require specialized mathematical knowledge not needed in other settings. . . .Teachers need to know mathematics in ways useful for, among other things, making mathematical sense of student work, and choosing powerful ways of representing the subject so that it is understandable to students."

(Ball et al., 2008, pp. 401-404)

Understanding the content that students are learning is necessary but insufficient for teachers because the work of teaching is not merely doing the math oneself. It is important for teachers to understand the content from students' perspectives, be aware of what might make a particular procedure or skill confusing, and be able to represent it in ways that make sense to students. This is important for teaching all students but is particularly important for advancing equity in mathematics learning.

Equitable teaching involves knowing different representations, what each is especially good for, and the mathematical differences among them. Sensitivity to language is another dimension of the high-leverage content that teachers need. When students talk in informal ways, teachers need to be able to consider how their language corresponds to formal mathematical terms and definitions. Teachers often have to adapt examples in their materials; doing so while preserving the mathematical focus requires a special kind of mathematical insight and reasoning. Posing mathematically strategic tasks and questions, interpreting student work, and communicating with students' families all require specialized ways of knowing and using math that others do not need. Being able to hear the thinking underlying students' mathematical ideas and to acknowledge their competence is vital to disrupting patterns of inequity that position students as "struggling," a pattern that disproportionately affects students of historically marginalized identities (Gresalfi et al., 2009). Supporting teachers to help students thrive mathematically requires that they have both curriculum-based professional development and opportunities for professional learning that provide practice with these specialized ways of knowing mathematics for teaching.

Key Take-Aways

1. The U.S. school mathematics curriculum is broad, and comprises a wide range of topics, applications, and skills. This breadth can take time away from a deliberate focus on high-leverage mathematical content.

2. Focusing students' learning on high-leverage mathematical content and practices is essential for their ongoing development, including their access to pathways that depend on mathematical knowledge and skill, their disposition and capability to use mathematics in everyday situations, and a positive sense of themselves as "math doers."

3. Teachers need specialized understanding of mathematics to help students thrive mathematically. Being able to discern and emphasize high-leverage mathematical content and practices depends on teachers' own mathematical knowledge for teaching.

4. Teachers need professional learning opportunities to continue to develop their mathematical knowledge for teaching.

Reflect & Analyze:

1. How do you react to situations involving mathematical reasoning—for instance, about how large something is or how to fit something into some space or to figure out how likely something is? How does your experience with mathematics in and out of school play out in situations like these? Did you experience math as bits and pieces of skills or did you develop a broader view that highlighted high-leverage concepts, skills, and practices?

2. What are particular concepts, skills, or practices of mathematics or mathematical reasoning you use in your life, whether formally or not? When do you feel competent in math and what contributes to that?

3. What are moments in teaching where you find yourself trying to explain something you know how to do but feel stuck about how to unpack it? What resources do you have to address those moments? What are moments when you see that you have the special mathematical understanding needed to help a student who is confused?

Key References

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