New York State Next Generation Mathematics Learning Standards Unpacking Document (DRAFT)

Course: Algebra II	**DOMAIN: Number and Quantity (N)-The Complex Number System**

CLUSTER: Perform arithmetic operations with complex numbers.

Students continue with their extension of number with the introduction of the imaginary number *i* and complex numbers (the sum of a real and imaginary number/ set of numbers of the form *a+bi* where i^2= -1 and *a* and *b* are real numbers). Students understand complex numbers as a superset of the real numbers (i.e., a complex number $a + bi$ is real when $b = 0$) that forms a number system. That is, you can add, subtract, multiply and divide two numbers of this form and get another number of the same form as the result. Students learn that complex numbers share many similar properties of the real numbers: associative, commutative, distributive and use these properties to simplify expressions involving complex numbers using multiple operations (addition, subtraction, and/or multiplication only).

Grade Level Standards:

AII-N.CN.1 Know there is a complex number *i* such that $i^2 = –1$, and every complex number has the form *a + bi* with *a* and *b* real.
AII-N.CN.2 Use the relation $i^2 = –1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
Note: Tasks include simplifying powers of *i*.

PERFORMANCE/KNOWLEDGE TARGETS
(measurable and observable)

- Simplify higher powers of *i* to either *i*, 1, -1, or –*i*.
- Rewrite and simplify radicals with a negative radicand in terms of *i*
- Simplify complex number expressions to simplest *a + bi* form using multiple operations (addition, subtraction, and/or multiplication; no division) and the commutative, associative, and distributive properties.

ASPECTS OF RIGOR

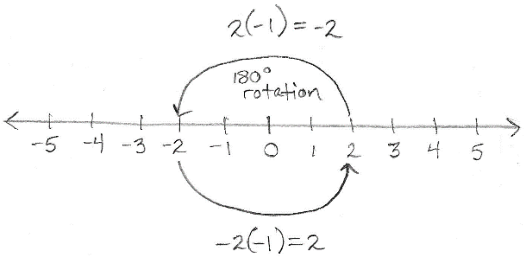
Procedural Conceptual Application

MATHEMATICAL PRACTICES	1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.
FOUNDATIONAL UNDERSTANDING	**AI-A.REI.4b** Solve quadratics by i) inspection ii) taking square roots iii) factoring iv) completing the square v) the quadratic formula, and vi) graphing. Recognize when the process yields no real solutions. (shared standard with Algebra I and II) **AI-N.RN.3** Use properties and operations to understand the different forms of rational and irrational numbers. **AI-N.RN.3a** Perform all four arithmetic operations and apply properties to generate equivalent forms of rational numbers and square roots. Tasks include rationalizing denominators.

The following pages contain EXAMPLES to support current instruction of the content standard and may be used at the discretion of the teacher and adapted to best serve the needs of the learners in the classroom.

One method in which students can develop an understanding of the imaginary number *i* is by utilizing prior knowledge of transformational geometry (scale factors and rotations). The following is taken from lesson 37 of Engage NY Algebra II, Module 1.

Recall that multiplying by -1 rotates the number line in the plane by $180°$ about the point 0.



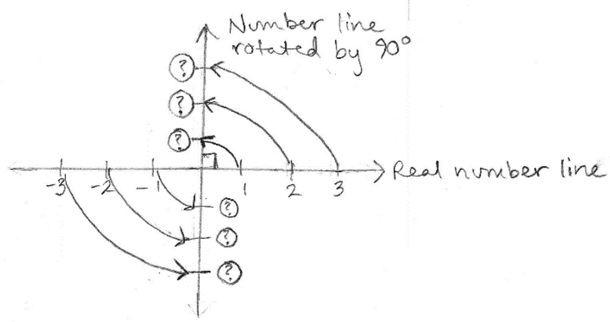
Think about the equation x^2=4.

$1 \cdot x \cdot x = 4$

Which transformation x, when applied two times in a row will turn a 1 into a 4? Scale by 2 or scale by -2. What if the equation was x^2= -4.

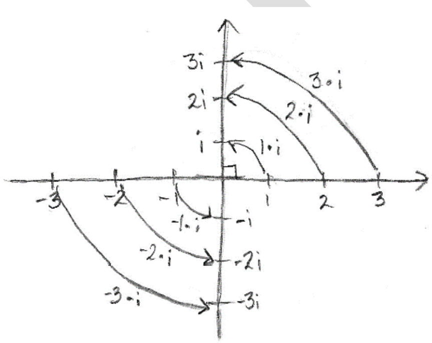
Is there a number we can multiply by that corresponds to a $90°$ rotation?

Such a number *does not* map the number line to itself, so we must *imagine* another number line that is a $90°$ rotation of the original:



This is like the coordinate plane. However, how should we label the points on the vertical axis?

Well, since we *imagined* such a number existed, let's call it the imaginary axis and subdivide it into units of something called i. Then, the point 1 on the number line rotates to $1 \cdot i$ on the rotated number line and so on, as follows:



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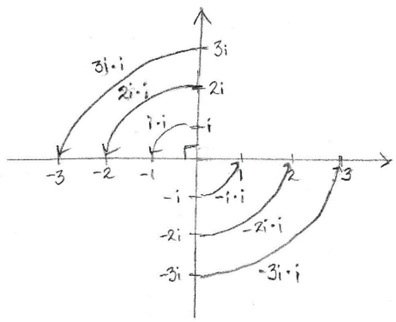
What happens if we multiply a point on the vertical number line by i? We rotate that point by 90°counterclockwise.



When we perform two $90°$ rotations, it is the same as performing a $180°$ rotation, so multiplying by i twice results in the same rotation as multiplying by -1. Since two rotations by $90°$ is the same as a single rotation by $180°$, two rotations by $90°$ is equivalent to multiplication by i twice, and one rotation by $180°$ is equivalent to multiplication by -1, we have

$$i^2 \cdot x = -1 \cdot x$$

for any real number x; thus,

$$i^2 = -1.$$

Why might this new number i be useful? It allows us to solve more equations.

Recall that there are no real solutions to the equation

$$x^2 + 1 = 0.$$

However, this new number i is a solution.

$$(i)^2 + 1 = -1 + 1 = 0$$

In fact, "solving" the equation $x^2 + 1 = 0$, we get

$$x^2 = -1$$

$$\sqrt{x^2} = \sqrt{-1}$$

$$x = \sqrt{-1} \text{ or } x = -\sqrt{-1}.$$

However, because we know that $i^2 = -1$, and $(-i)^2 = (-1)^2(i)^2 = -1$, we have two solutions to the quadratic equation $x^2 = -1$, which are i and $-i$.

These results suggest that $i = \sqrt{-1}$.

For further examples and an alternative additional algebraic interpretation for imaginary numbers, see Intro to the Imaginary Numbers , available free from Khan Academy, 2019 (Algebra II, Complex Numbers, What are the Imaginary Numbers) . The following is taken from the section "Why do we have imaginary numbers anyway?"

The answer is simple. The imaginary unit *i* allows us to find solutions to many equations that do not have real number solutions. This may seem weird, but it is actually very common for equations to be unsolvable in one number system but solvable in another, more general number system. Here are some examples with which you might be more familiar.

- With only the counting numbers, we can't solve x+8=1; we need the integers for this!
- With only the integers, we can't solve 3x-1=0; we need the rational numbers for this!
- With only the rational numbers, we can't solve x^2=2. Enter the irrational numbers and the real number system!

And so, with only the real numbers, we can't solve x^2= -1. We need the imaginary numbers for this!

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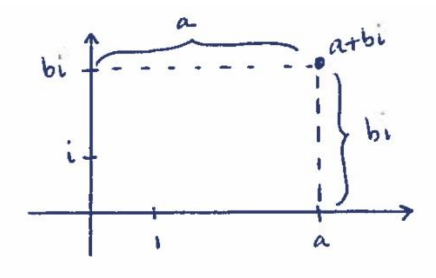
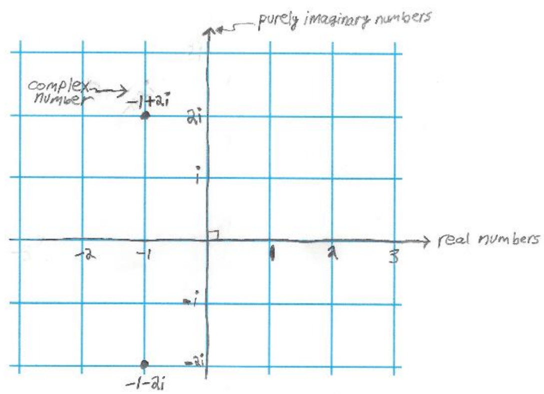
The Complex Plane and the Complex Number System

Students can build a graphic organizer (i.e., Venn Diagram) to see the relationship of the various numbers in the complex number system (complex, imaginary, rational, irrational, integers, whole and natural). What type of numbers are purely real, purely imaginary?





Name a few complex numbers for students to plot on their graph paper. This builds an understanding of their locations in this coordinate system. For example, consider $-2i-3$, $-i$, i, $i-1$, and $\frac{3}{2}i+2$. Make sure students are also cognizant of the fact that real numbers are also complex numbers (e.g., $-\frac{3}{2}$, 0, 1, π).

All complex numbers can be written in the form

$$a+bi,$$

where a and b are real numbers. Just as we can represent real numbers on the number line, we can represent complex numbers in the complex plane. Each complex number $a+bi$ can be located in the complex plane in the same way we locate the point (a,b) in the Cartesian plane. From the origin, translate a units horizontally along the real axis and b units vertically along the imaginary axis.

Complex numbers are built from real numbers, we should be able to add, subtract, multiply, and divide them. They should also satisfy the commutative, associative, and distributive properties, just as real numbers do.

Algebra II expectations do not include representing complex numbers and their operations on the complex plane, however students can extend their work with arithmetic operations with complex numbers through a geometric point of view (transformations) to answer what happens when you

- Multiply a real number by a complex number
- Multiply a complex number by *i*

Students can also use prior knowledge of vectors (grade 8 and geometry) to see geometrically how to add/subtract complex numbers. See lessons 5 and 6 in [Engage NY Pre-Calculus Module 1] for possible instruction ideas. Students should see that the properties of real numbers (commutative, associative and distributive) apply to the complex numbers as well.

Area Models may be used to show the multiplication of complex numbers, for example:

	8	$7i$
10	80	$70i$
$-5i$	$-40i$	$-35i^2$

(8+7*i*) (10-5*i*) = 115 +30*i*

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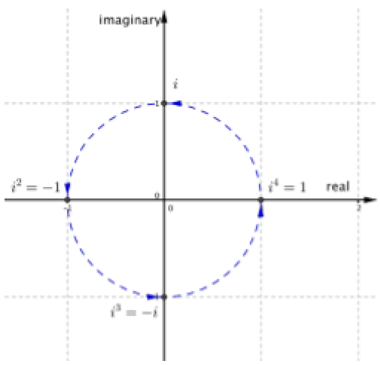
Examples of operations involving complex numbers:

1. Express in simplest *a + bi* form: $\left(8-\sqrt{-50}\right)\left(-2+\sqrt{-18}\right) = 14+34\sqrt{2}i$

2. Express in simplest *a + bi* form: $(4+3xi)(1+2i)-(4-3xi)(1+2i) = -12x+6xi$

3. Express in simplest form: $17xi^{42}+12xi^{89}-14xi^{63}=-17x+26xi$



For evaluating powers of *i*, students should see the pattern or the cycle of every 4th rotation, $i^4=1$.

Students will be extending work with complex numbers when investigating the types of solutions (and graphs) that polynomial equations (functions) have, connecting work with standards AII-A.REI.4b and AII-F.IF.7c.